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Some duality relations in the theory of tensor products

Petr Hájek[a,1], Richard J. Smith[b,*]

[a] *Mathematical Institute, Czech Academy of Science, Žitná 25, 115 67 Praha 1, Czech Republic*
[b] *School of Mathematical Sciences, University College Dublin, Belfield, Dublin 4, Ireland*

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Abstract

We study several classical duality results in the theory of tensor products, due mostly to Grothendieck, providing new proofs as well as new results. In particular, we show that the canonical mapping $Y^* \otimes_\pi X \to (\mathcal{L}(X, Y), \tau)^*$, where τ is the topology of uniform convergence on compact subsets of X, is not always injective. This answers negatively a problem of Defant and Floret. We use the machinery of vector measures to give new proofs of the dualities $(X \otimes_\varepsilon Y)^* = \mathcal{N}(X, Y^*)$, whenever Y^* has the Radon–Nikodým property, and (a slight improvement of) a result of Rosenthal: $(X \otimes_\varepsilon Y)^* \subset \overline{\mathcal{F}}(X, Y^*)$ whenever $\ell_1 \not\hookrightarrow Y$. Here, $\mathcal{N}(X, Y^*)$ and $\mathcal{F}(X, Y^*)$ denote the spaces of nuclear and finite rank operators from X to Y^*, respectively.

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1. Introduction and preliminaries

An important result in the topological theory of tensor products is the theorem of Grothendieck, which describes the linear topological dual of the space of bounded linear operators $\mathcal{L}(X, Y)$, equipped with the topology τ of uniform convergence on compact

* Corresponding author. Tel.: +353 17162570.
E-mail addresses: hajek@math.cas.cz (P. Hájek), richard.smith@ucd.ie (R.J. Smith).
[1] Tel.: +420 222090705.

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subsets of X. According to this result, the continuous linear functionals on $(\mathcal{L}(X, Y), \tau)$ consist of all ϕ of the form

$$\phi(T) = \sum_{i=1}^{\infty} \langle y_i^*, Tx_i \rangle, \quad \text{where } x_i \in X, y_i^* \in Y^* \text{ satisfy } \sum_{i=1}^{\infty} \|x_i\| \, \|y_i^*\| < \infty. \tag{1}$$

This formulation of Grothendieck's theorem is taken from [12, Proposition 1.e.3]. Its advantage is that it uses only elementary functional analytic language. However, it is more natural to rephrase this result using the language of tensor products: the canonical mapping $Y^* \otimes_\pi X \to (\mathcal{L}(X, Y), \tau)^*$, described by formula (1), is surjective. This is the formulation to be found in [2, Proposition 5.5]. A natural question posed, for example, by Defant and Floret [2, p. 65], is whether the canonical mapping above is also injective. In Theorem 2.5 we give a negative solution to this problem. We proceed by giving a new proof of the classical duality result $(X \otimes_\varepsilon Y)^* = \mathcal{N}(X, Y^*)$, whenever Y^* has the Radon–Nikodým property (RNP). Our proof avoids the machinery of integral operators and uses instead the theory of Bochner integration. A recent result of Rosenthal [14] (combined with a result of John [10]) asserts that every integral operator from X to Y^* is compact (in fact, approximable) if and only if Y does not contain a copy of ℓ_1. We use the theory of Pettis integration to provide a direct proof of this fact. In the final section, we give a new proof of another celebrated result of Grothendieck, which states that X^* has the metric approximation property whenever it is a RNP space with the approximation property (AP).

We begin by collecting some basic definitions and results, the proofs of which may be found in [3,2,12,15,5]. Our notation is standard. By $\mathcal{F}(X, Y)$ we denote the space of all finite rank operators from $\mathcal{L}(X, Y)$. The space $\mathcal{F}_{w^*}(X^*, Y)$ of all w^*-w continuous operators in $\mathcal{F}(X^*, Y)$ identifies naturally with the algebraic tensor product $X \otimes Y$. We denote the space of all integral operators by $\mathcal{I}(X, Y)$. We recall that the couple $\langle \mathcal{L}(X, Y^*), X \otimes Y \rangle$ forms a duality pair: given $T \in \mathcal{L}(X, Y^*)$ and $z = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$, put

$$\langle T, z \rangle = \sum_{i=1}^{n} \langle T(x_i), y_i \rangle. \tag{2}$$

The pairing enables us to introduce the projective norm π on $X \otimes Y$:

$$\pi(z) = \sup\{\langle T, z \rangle, \|T\| \le 1, T \in \mathcal{L}(X, Y^*)\}.$$

The projective tensor product $X \otimes_\pi Y$ is the completion of $(X \otimes Y, \pi)$. Every element $z \in X \otimes_\pi Y$ admits a representation $z = \sum_{i=1}^{\infty} x_i \otimes y_i$, such that $\sum_{i=1}^{\infty} \|x_i\| \, \|y_i\| < \infty$ (where, without loss of generality, $\{\|x_i\|\}_{i=1}^{\infty} \in c_0$ and $\{\|y_i\|\}_{i=1}^{\infty} \in \ell_1$) and

$$\pi(z) = \inf \left\{ \sum_{i=1}^{\infty} \|x_i\| \, \|y_i\| : z = \sum_{i=1}^{\infty} x_i \otimes y_i \right\}.$$

The dual of the projective tensor product is described by the next result.

Proposition 1.1 (*[2, Chapter 3]*). *Let X, Y be Banach spaces. Then the canonical dual pairing gives the linear topological duality*

$$(X \otimes_\pi Y)^* = \mathcal{L}(X, Y^*).$$

Closely connected to the projective tensor product $X^* \otimes_\pi Y$ is the notion of the nuclear operator. An operator $T : X \to Y$ is called nuclear if there exists a pair of sequences $\{x_i^*\}_{i=1}^\infty$ in X^* and $\{y_i\}_{i=1}^\infty$ in Y such that $\sum_{i=1}^\infty \|x_i^*\| \, \|y_i\| < \infty$ and $Tx = \sum_{i=1}^\infty \langle x_i^*, x\rangle y_i$. The nuclear norm is defined by

$$N(T) = \inf \left\{ \sum_{i=1}^\infty \|x_i^*\| \, \|y_i\| : T(x) = \sum_{i=1}^\infty \langle x_i^*, x\rangle y_i \right\}.$$

The Banach space of nuclear operators is denoted by $\mathcal{N}(X, Y)$, or $\mathcal{N}(X)$ if $Y = X$. The formal identity $J : X^* \otimes_\pi Y \to \mathcal{N}(X, Y)$, given by $J(\sum_{i=1}^\infty x_i^* \otimes y_i) = \sum_{i=1}^\infty x_i^* \otimes y_i$, for all pairs of sequences $\{x_i^*\}_{i=1}^\infty \in X^*, \{y_i\}_{i=1}^\infty \in Y$, such that $\sum_{i=1}^\infty \|x_i^*\| \, \|y_i\| < \infty$, is a well-defined quotient mapping. Let τ be the locally convex topology on $\mathcal{L}(X, Y)$ of uniform convergence on compact sets in X, generated by the seminorms $\|T\|_K$, where $K \subset X$ is norm compact.

Theorem 1.2 (*[7]*). *Let X be a Banach space. The following conditions are equivalent.*

(i) *X has the AP.*
(ii) *For every Banach space Y, $\overline{\mathcal{F}}^\tau(X, Y) = \mathcal{L}(X, Y)$.*
(iii) *For every Banach space Y, $\overline{\mathcal{F}}^\tau(Y, X) = \mathcal{L}(Y, X)$.*
(iv) *$J : X^* \otimes_\pi X \to \mathcal{N}(X)$ is injective or, equivalently, it is an isometry.*
(v) *For every Banach space Y, $J : Y^* \otimes_\pi X \to \mathcal{N}(Y, X)$ is injective or, equivalently, it is an isometry.*
(vi) *For every Banach space Y, every $T \in \mathcal{K}(Y, X)$ and $\varepsilon > 0$, there exists $F \in \mathcal{F}(Y, X)$ with $\|T - F\| < \varepsilon$.*

The next theorem is almost certainly known to specialists. As we have not found an explicit reference, we include its proof for the convenience of the reader.

Theorem 1.3. *Let X be a Banach space. The following conditions are equivalent.*

(i) *X^* has the AP.*
(ii) *$J : X^* \otimes_\pi X^{**} \to \mathcal{N}(X, X^{**})$ is injective or, equivalently, an isometry.*
(iii) *For every Banach space Y, $J : X^* \otimes_\pi Y \to \mathcal{N}(X, Y)$ is an isometry.*

Proof. (ii) $\Rightarrow$ (i). It is well-known [9, p. 326] that the formal transposition mapping $t : E \otimes_\pi F \to F \otimes_\pi E, t(\sum_{i=1}^\infty e_i \otimes f_i) = \sum_{i=1}^\infty f_i \otimes e_i$, is an isometric linear isomorphism. Next, we note that $\mathcal{N}(X, X^{**})$ and $\mathcal{N}(X^*, X^*)$ are canonically isometric, via the transposition of their elements $z = \sum_{i=1}^\infty x_i^* \otimes x_i^{**} \leftrightarrow z' = \sum_{i=1}^\infty x_i^{**} \otimes x_i^*$. Indeed, $\mathcal{N}(X, X^{**})$ is a quotient (via J) of $X^* \otimes_\pi X^{**}$, while $\mathcal{N}(X^*, X^*)$ is a quotient (via J') of the isometric transpose $t(X^* \otimes_\pi X^{**}) = X^{**} \otimes_\pi X^*$. The kernels are described as follows.

$$\ker J = \left\{ z = \sum_{i=1}^\infty x_i^* \otimes x_i^{**} : \sum_{i=1}^\infty x_i^*(x) x_i^{**} = 0 \text{ for all } x \in X \right\}$$

$$\text{and} \quad \ker J' = \left\{ z' = \sum_{i=1}^{\infty} x_i^{**} \otimes x_i^* : \sum_{i=1}^{\infty} x_i^{**}(x^*) x_i^* = 0 \text{ for all } x^* \in X^* \right\}.$$

Both of these conditions are indeed equivalent to the single condition $z \in \ker J$ if and only if $t(z) \in \ker J'$, which is to say $\sum_{i=1}^{\infty} x_i^{**}(x^*) x_i^*(x) = 0$ for all $x \in X, x^* \in X^*$. Using the transposition we may transform (ii) of Theorem 1.3 into the equivalent statement that $J' : X^{**} \otimes_\pi X^* \to \mathcal{N}(X^*, X^*)$ is an isometry. By condition (iv) of Theorem 1.2, we conclude that X^* has the AP.

(iii) $\Rightarrow$ (ii) is immediate. It remains to show (i) $\Rightarrow$ (iii). Let $z = \sum_{i=1}^{\infty} x_i^* \otimes y_i \in X^* \otimes_\pi Y$ be non-zero; our goal is to show that $J(z) \neq 0$. Without loss of generality, we may assume that $\sum_{i=1}^{\infty} \|y_i\| < \infty$ and $\lim_{i \to \infty} \|x_i^*\| = 0$. We proceed by contradiction, assuming that $J(z)(x) = \sum_{i=1}^{\infty} x_i^*(x) y_i = 0$ for all $x \in X$. The assumptions imply $\sum_{i=1}^{\infty} x^{**}(x_i^*) y_i = 0$ for all $x^{**} \in X^{**}$. Given $\varepsilon > 0$, as X^* has the AP, there is

$$F = \sum_{k=1}^{n} u_k^{**} \otimes u_k^* \in \mathcal{F}(X^*),$$

such that $\sup_i \|F(x_i^*) - x_i^*\| < \varepsilon$. We let $z' = \sum_{i=1}^{\infty} F(x_i^*) \otimes y_i \in X^* \otimes_\pi Y$. Note the important fact that $z' \in X^* \otimes Y$ is actually a finite tensor. Indeed,

$$z' = \sum_{i=1}^{\infty} \left(\sum_{k=1}^{n} u_k^{**}(x_i^*) u_k^* \right) \otimes y_i = \sum_{k=1}^{n} u_k^* \otimes \left(\sum_{i=1}^{\infty} u_k^{**}(x_i^*) y_i \right).$$

Next, $J(z')$ satisfies

$$J(z')(x) = \sum_{i=1}^{\infty} \langle F(x_i^*), x \rangle y_i = \sum_{i=1}^{\infty} F^*(x)(x_i^*) y_i = 0, \quad \text{for every } x \in X.$$

Hence $J(z') = 0$, as an element of $\mathcal{N}(X, Y)$, and since z' is also a finite tensor we conclude that $z' = 0$ as an element of $X^* \otimes_\pi Y$. Hence we have an estimate

$$\pi(z) = \pi(z - z') = \pi \left(\sum_{i=1}^{\infty} x_i^* \otimes y_i - \sum_{i=1}^{\infty} F(x_i^*) \otimes y_i \right) \leq \varepsilon \sum_{i=1}^{\infty} \|y_i\|.$$

Since ε was arbitrarily small, we conclude that $\pi(z) = 0$ as desired. According to the definitions of the projective and nuclear norms, it follows that J is moreover an isometry. $\square$

2. The dual of $(\mathcal{L}(X, Y), \tau)$

Denote by $i : \mathcal{L}(X, Y) \to \mathcal{L}(X, Y^{**})$ the formal identity embedding. By Proposition 1.1 (and the transposition isometry $Y^* \otimes_\pi X = t(X \otimes_\pi Y^*)$) we have

$$(Y^* \otimes_\pi X)^* = \mathcal{L}(X, Y^{**}) = \mathcal{L}(Y^*, X^*).$$

We consider the w^*-topology on $\mathcal{L}(X, Y^{**})$, or $\mathcal{L}(Y^*, X^*)$, originating from this duality. Then we have the following.

Lemma 2.1. *The mapping*

$$i : (\mathcal{L}(X, Y), \tau) \to (\mathcal{L}(X, Y^{**}), w^*)$$

is continuous. In particular, the (restriction of the) dual mapping

$$i^* : Y^* \otimes_\pi X \to (\mathcal{L}(X, Y), \tau)^* \tag{3}$$

is w-w^* continuous (the topologies come from the duality pairs described above).

Proof. Every $z \in Y^* \otimes_\pi X$ admits a representation $z = \sum_{i=1}^\infty y_i^* \otimes x_i$, with the property that $\{\|x_i\|\}_{i=1}^\infty \in c_0$ and $\{\|y_i^*\|\}_{i=1}^\infty \in \ell_1$. Let $K = \{x_i\}_{i=1}^\infty \cup \{0\}$, and let U be the τ-open set in $\mathcal{L}(X, Y)$ defined by $U = \{T : \sup_{x \in K} \|T(x)\| < 1\}$. Clearly, $T \in U$ implies $|y^*(T(x))| < \|y^*\|$ for all $y^* \in Y^*$, $x \in K$. Thus $|\langle T, \sum_{i=1}^\infty y_i^* \otimes x_i\rangle| \le \sum_{i=1}^\infty \|y_i^*\| < \infty$ for all $T \in U$, which finishes the proof. The second result follows by duality. $\square$

Let us call $\mathcal{T} : \mathcal{L}(X, Y) \to \mathcal{L}(Y^*, X^*), \mathcal{T}(T) = T^*$, the conjugation operator. Of course, $\mathcal{T}$ is an isometric embedding whose target space is a dual space canonically isometric to $\mathcal{L}(X, Y^{**})$.

Proposition 2.2. *Let X, Y be Banach spaces. Then $\mathcal{L}(X, Y) \subseteq \mathcal{L}(X, Y^{**})$ (or $\mathcal{T}(\mathcal{L}(X, Y)) \subseteq \mathcal{L}(Y^*, X^*)$), is w^*-dense if and only if i^* is injective.*

Proof. $\mathcal{L}(X, Y)$ is w^*-dense in $\mathcal{L}(X, Y^{**})$ if and only if $z \in Y^* \otimes_\pi X$ is zero whenever $z \in \mathcal{L}(X, Y)^\perp$ or, equivalently, $z = 0$ whenever $i^*(z) = 0$. The second case follows by standard transposition. $\square$

The following is a more complete formulation of Grothendieck's duality result.

Theorem 2.3 (*Grothendieck, [2, Proposition 5.5]*)**.** *The map i^* from (3) is surjective. In other words, the continuous linear functionals on $(\mathcal{L}(X, Y), \tau)$ consist of all ϕ of the form*

$$\phi(T) = \sum_{i=1}^\infty \langle y_i^*, Tx_i\rangle, \quad \textit{where } x_i \in X, y_i^* \in Y^* \textit{ satisfy } \sum_{i=1}^\infty \|x_i\|\,\|y_i^*\| < \infty.$$

In some cases, the mapping i^* is injective.

Theorem 2.4 (*[2, p. 65]*)**.** *Let X, Y be Banach spaces. Suppose that either X or Y^* has the AP, or that Y is reflexive. Then the mapping $i^* : Y^* \otimes_\pi X \to (\mathcal{L}(X, Y), \tau)^*$ from (3) is injective. In particular, we may write $(\mathcal{L}(X, Y), \tau)^* = Y^* \otimes_\pi X$. The pairing is canonical:*

$$\langle z, T\rangle = \sum_{i=1}^\infty \langle y_i^*, Tx_i\rangle, \quad T \in \mathcal{L}(X, Y), z = \sum_{i=1}^\infty y_i^* \otimes x_i \in Y^* \otimes_\pi X.$$

Our first main result is contained in the next characterization.

Theorem 2.5. *Let Y be a Banach space with the AP. The following conditions are equivalent.*

(i) *Y^* has the AP.*

(ii) *For every Banach space* X*, the map* $i^* : Y^* \otimes_\pi X \to (\mathcal{L}(X, Y), \tau)^*$ *from* (3) *is injective.*

(iii) $i^* : Y^* \otimes_\pi Y^{**} \to (\mathcal{L}(Y^{**}, Y), \tau)^*$ *is injective.*

Proof. The implication (ii) $\Rightarrow$ (iii) is trivial. We prove (iii) $\Rightarrow$ (i). By using Theorem 1.3, it suffices to show that $J : Y^* \otimes_\pi X \to \mathcal{N}(Y, X)$ is an isometry. Recall that

$$\ker i^* = \left\{ z = \sum_{i=1}^{\infty} y_i^* \otimes x_i : \langle z, S\rangle = \sum_{i=1}^{\infty} \langle y_i^*, S(x_i)\rangle = 0, \right.$$
$$\left. \text{for all } S \in \mathcal{L}(X, Y) \right\}. \tag{4}$$

As Y is assumed to have the AP, we have by condition (iii) in Theorem 1.2 that, for every X, $\overline{\mathcal{F}}^{\tau}(X, Y) = \mathcal{L}(X, Y)$. Thus by the bipolar and Hahn–Banach theorems, (4) is equivalent to the next condition.

$$\ker i^* = \left\{ z = \sum_{i=1}^{\infty} y_i^* \otimes x_i : \langle z, S\rangle = \sum_{i=1}^{\infty} \langle y_i^*, S(x_i)\rangle = 0, \right.$$
$$\left. \text{for all } S \in \mathcal{F}(X, Y) \right\}. \tag{5}$$

Now compare this condition with that which describes the kernel of J:

$$\ker J = \left\{ z = \sum_{i=1}^{\infty} y_i^* \otimes x_i : \langle T, z\rangle = \sum_{i=1}^{\infty} \langle T(y_i^*), x_i\rangle = 0, \right.$$
$$\left. \text{for all } T \in \mathcal{F}_{w^*}(Y^*, X^*) \right\}. \tag{6}$$

We claim that (5) and (6) are equivalent conditions. Indeed, it suffices to note that taking the adjoints $S \to S^*$ induces an isometry from $\mathcal{F}(X, Y)$ onto $\mathcal{F}_{w^*}(Y^*, X^*)$, so we can reformulate (5) as

$$\ker i^* = \left\{ z = \sum_{i=1}^{\infty} y_i^* \otimes x_i : \langle z, S\rangle = \sum_{i=1}^{\infty} \langle S^*(y_i^*), x_i\rangle = 0, \text{ for all } S \in \mathcal{F}(X, Y) \right\}$$

which is precisely (6). Since i^* is assumed to be injective, so is J. As above, by the definitions of the projective and nuclear norms, J is an isometry. This proves that, indeed, Y^* has the AP. The implication (i) $\Rightarrow$ (ii) follows from Theorem 2.4. $\square$

There exist Banach spaces with the AP whose dual fails the AP. The construction of such spaces relies of course on the fundamental result of Enflo [4], which is presented, for example, in [12, Theorem 1.e.7.] (using the method of [8,11]). Alternatively, one can use the space constructed in [6]. Therefore, we obtain a negative solution to the problem of Defant and Floret.

3. Duality of the injective tensor product $X \otimes_\varepsilon Y$

In this section, we are going to investigate the Banach space dual to the injective tensor product space $X \otimes_\varepsilon Y$. The injective norm on $X \otimes Y$ agrees with the operator norm on $\mathcal{F}_{w^*}(X^*, Y)$, thus we can identify $X \otimes_\varepsilon Y$ with the norm closure of $\mathcal{F}_{w^*}(X^*, Y)$ in $\mathcal{L}(X^*, Y)$. The following result is fundamental.

Theorem 3.1 (*Grothendieck*)**.** *Let X, Y be Banach spaces. There is an isometry*

$$(X \otimes_\varepsilon Y)^* = \mathcal{I}(X, Y^*).$$

The notion of integral operators is rather abstract. We are going to investigate two special cases of the above theorem, namely the case when Y is an Asplund space (equivalently, Y^* has the RNP) and the more general case when $\ell_1 \not\hookrightarrow Y$. Our approach is to use the theory of vector integration (in the sense of Bochner and Pettis, respectively) to obtain new proofs and new results, giving a more concrete description of $\mathcal{I}(X, Y^*)$. We refer the reader to [3,15] for definitions and background on Bochner and Pettis integration. The dual balls B_{X^*}, B_{Y^*} are assumed to be equipped with the w^*-topology, unless specified otherwise. We will rely on the following results.

Theorem 3.2 (*Schwartz [1, Corollary 7.8.7]*)**.** *Let X be an Asplund space. Then for every w^*-Radon measure μ on B_{X^*}, $Id : B_{X^*} \to B_{X^*}$ is μ-Bochner integrable.*

The next result follows from [15, Corollary 7-3-8], which Talagrand attributes to Musial and Janicka.

Theorem 3.3 (*Musial and Janicka [15, Corollary 7-3-8]*)**.** *Assume that $\ell_1 \not\hookrightarrow X$. Then for every w^*-Radon measure μ on B_{X^*}, $Id : B_{X^*} \to B_{X^*}$ is μ-Pettis integrable.*

Corollary 3.4. *Let Y be an Asplund space or suppose $\ell_1 \not\hookrightarrow Y$, respectively. Let X be an arbitrary Banach space and μ a Radon measure on $(B_{X^*}, w^*) \times (B_{Y^*}, w^*)$. Then $I : B_{X^*} \times B_{Y^*} \to Y^*$, $I(x^*, y^*) = y^*$, is μ-Bochner integrable, or μ-Pettis integrable, respectively.*

Proof. Let $P : X^* \times Y^* \to Y^*$ be the (w^*-continuous) projection. Denote the image measure $P\mu$ by η. Since $Id : B_{Y^*} \to B_{Y^*}$ is η-Bochner integrable by Theorem 3.2, there exist simple functions $f_n : Y^* \to Y^*$ such that $\lim_{n\to\infty} \int_{B_{Y^*}} \|f_n - Id\| \, \mathrm{d}\eta = 0$. Let $g_n : B_{X^*} \times B_{Y^*} \to Y^*$ denote the simple functions $f_n \circ P$. Clearly, $\lim_{n\to\infty} \int_{B_{X^*} \times B_{Y^*}} \|g_n - I\| \, \mathrm{d}\mu = 0$, so I is μ-Bochner integrable.

The Pettis case is similar. By the same argument with compositions, we see that I is μ-Dunford integrable. Given any w^*-Borel set $E \subset B_{X^*} \times B_{Y^*}$, $\eta = P(\mu \restriction_E)$ is a Radon measure on B_{Y^*}. The η-Dunford integral satisfies $\int_{B_{Y^*}} Id \, \mathrm{d}\eta \in Y^*$, and so $\int_E I \, \mathrm{d}\mu = \int_{B_{Y^*}} Id \, \mathrm{d}\eta \in Y^*$. Thus I is μ-Pettis integrable by definition. $\square$

We also need two facts to be found, for example, in [3].

Lemma 3.5 (*[3, Lemma VI.3]*)**.** *Let (S, Σ, μ) be a finite positive measure space and $f : S \to X$ be Bochner integrable. For each $\varepsilon > 0$ there are sequences $\{x_n\}_{n=1}^\infty$ in X*

and $\{E_n\}_{n=1}^{\infty}$ in Σ (not necessarily pairwise disjoint), such that

$$\sum_{n=1}^{\infty} \chi_{E_n} x_n \text{ converges to } f \text{ absolutely } \mu\text{-a.e.}$$

$$\text{and} \quad \int \|f\| d\mu - \varepsilon \le \sum_{n=1}^{\infty} \|x_n\| \mu(E_i) \le \int \|f\| \mathrm{d}\mu + \varepsilon.$$

Lemma 3.6 ([3, Theorem VIII.5]). *There is a canonical isometric embedding*

$$j : \overline{\mathcal{F}_{w^*}}(X^*, Y) = X \otimes_\varepsilon Y \hookrightarrow C(B_{X^*} \times B_{Y^*})$$

given by

$$j(S)(x^*, y^*) = \langle y^*, S(x^*) \rangle.$$

The essential point of Theorem 3.1 is stated in the following theorem.

Theorem 3.7 (Grothendieck). *Let $j : X \otimes_\varepsilon Y \hookrightarrow C(B_{X^*} \times B_{Y^*})$ be the isometric embedding from Lemma 3.6. Then every $\phi \in (X \otimes_\varepsilon Y)^*$ has a representation as a positive w^*-Radon measure μ on $(B_{X^*} \times B_{Y^*}, w^* \times w^*)$, so that for $z \in X \otimes_\varepsilon Y$*

$$\langle \phi, z \rangle = \int_{B_{X^*} \times B_{Y^*}} j(z)(x^*, y^*) \mathrm{d}\mu = \int_{B_{X^*} \times B_{Y^*}} \langle x^* \otimes y^*, z \rangle \mathrm{d}\mu. \tag{7}$$

Moreover, $\|\phi\| = |\mu|(B_{X^*} \times B_{Y^*})$.

We now proceed with a new proof of the following classical duality theorem of Grothendieck.

Theorem 3.8 (Grothendieck). *Let Y^* be a RNP space. Then there is an isometry*

$$(X \otimes_\varepsilon Y)^* = \mathcal{N}(X, Y^*). \tag{8}$$

More precisely, every $\phi \in (X \otimes_\varepsilon Y)^*$, $\|\phi\| < 1$, is represented by a nuclear operator $T \in \mathcal{N}(X, Y^*)$, $T = \sum_{n=1}^{\infty} x_n^* \otimes y_n^*$, $\sum_{n=1}^{\infty} \|x_n^*\| \|y_n^*\| < 1$, so that for every $S \in \overline{\mathcal{F}_{w^*}}(X^*, Y) = X \otimes_\varepsilon Y$ we have

$$\langle T, S \rangle = \sum_{n=1}^{\infty} \langle y_n^*, S(x_n^*) \rangle. \tag{9}$$

Proof. By Theorem 3.7, every $\phi \in (X \otimes_\varepsilon Y)^*$, $\|\phi\| < 1$, is represented by a positive w^*-Radon measure μ on $B_{X^*} \times B_{Y^*}$, $|\mu| < 1$. Our goal is to represent ϕ as a nuclear operator $T \in \mathcal{N}(X, Y^*)$. We are going to define T by using the commutative diagram:

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "i_1"] & Y^* \\ C(B_{X^*} \times B_{Y^*}) \arrow[r, "i_2"] & L_1(\mu) \arrow[u, "i_3"] \end{tikzcd}$$

where $i_1(x)(x^*, y^*) = x^*(x)$, i_2 is the formal identity mapping and i_3 is defined by the formula

$$i_3(f) = \int_{B_{X^*} \times B_{Y^*}} f(x^*, y^*) y^* \, \mathrm{d}\mu.$$

Clearly, $\|i_1\| = 1$ and $\|i_2\| = |\mu| < 1$. The integrand in the definition of i_3 is a product of an integrable scalar function with the mapping $(x^*, y^*) \mapsto y^*$. Due to Corollary 3.4, the latter is μ-Bochner integrable. Again, we have $\|i_3\| < 1$. Thus $T = i_3 \circ i_2 \circ i_1$ is well-defined. Next, we claim that the linear operator $i_3 \circ i_2 : C(B_{X^*} \times B_{Y^*}) \to Y^*$ is nuclear. Using Lemma 3.5, for $\varepsilon > 0$ small enough, there are sequences $\{y_n\}_{n=1}^{\infty}$ in Y^* and $\{E_n\}_{n=1}^{\infty}$ of w^*-Borel subsets of $B_{X^*} \times B_{Y^*}$, so that

$$\int \|y^*\| \mathrm{d}\mu - \varepsilon \le \sum_{n=1}^{\infty} \|y_n^*\| \mu(E_i) \le \int \|y^*\| \mathrm{d}\mu + \varepsilon < 1 \tag{10}$$

and moreover

$$\begin{aligned}(i_3 \circ i_2)(f) &= \int_{B_{X^*} \times B_{Y^*}} f(x^*, y^*) y^* \mathrm{d}\mu \\ &= \int_{B_{X^*} \times B_{Y^*}} f(x^*, y^*) \sum_{n=1}^{\infty} \chi_{E_n} y_n^* \mathrm{d}\mu = \sum_{n=1}^{\infty} \left(\int_{E_n} f \, \mathrm{d}\mu \right) y_n^*. \end{aligned} \tag{11}$$

Note that $l_n(f) = \int_{E_n} \mathrm{d}\mu$ defines $l_n \in C(B_{X^*} \times B_{Y^*})^*$, $\|l_n\| = \mu(E_n)$. By (10), we see that $i_3 \circ i_2 = \sum_{n=1}^{\infty} l_n \otimes y_n^*$ is a nuclear operator with $N(i_3 \circ i_2) < 1$. Therefore, putting $x_n^* = i_1^*(l_n)$, we find that $T = \sum_{n=1}^{\infty} x_n^* \otimes y_n^*$ is a nuclear operator of norm less than one. Eq. (11) yields

$$T(x) = \int x(x^*, y^*) y^* \mathrm{d}\mu = \int x^*(x) y^* \mathrm{d}\mu = \sum_{n=1}^{\infty} x_n^*(x) y_n^*. \tag{12}$$

Given $z = \sum_{i=1}^{k} u_i \otimes v_i \in X \otimes Y$, by (7) and (12)

$$\begin{aligned}\langle \phi, z \rangle &= \int_{B_{X^*} \times B_{Y^*}} \sum_{i=1}^{k} y^*(v_i) x^*(u_i) \mathrm{d}\mu \\ &= \sum_{i=1}^{k} \left\langle \int_{B_{X^*} \times B_{Y^*}} x^*(u_i) y^* \mathrm{d}\mu, v_i \right\rangle \\ &= \sum_{i=1}^{k} \langle T(u_i), v_i \rangle = \sum_{n=1}^{\infty} \sum_{i=1}^{k} x_n^*(u_i) y_n^*(v_i) \\ &= \sum_{n=1}^{\infty} \left\langle y_n^*, \sum_{i=1}^{k} (u_i \otimes v_i)(x_n^*) \right\rangle = \langle T, z \rangle,\end{aligned}$$

and (9) follows. □

Our next theorem improves [14, Theorem 1], where the second condition reads instead $\mathcal{I}(X, Y^*) \subset \mathcal{K}(X, Y^*)$ for every Banach space X.

Theorem 3.9. *Let Y be a Banach space. The following conditions are equivalent.*

(i) $\ell_1 \not\hookrightarrow Y$
(ii) $\mathcal{I}(X, Y^*) \subset \overline{\mathcal{F}}(X, Y^*)$ *for every Banach space X.*

Proof. (i) $\Rightarrow$ (ii). If $T : X \to Y^*$ is an integral then so is its corresponding bilinear form on $X \times Y$. Thus there exists a Radon measure μ on $B_{X^*} \times B_{Y^*}$ with the property that

$$(Tx)(y) = \int_{B_{X^*} \times B_{Y^*}} x^*(x) y^*(y) \mathrm{d}\mu, \quad x \in X,\ y \in Y.$$

We can write $T = D \circ S$, where $S : X \to C(B_{X^*} \times B_{Y^*})$ is given by $(Sx)(x^*, y^*) = x^*(x)$, and $D : L_\infty(\mu) \to Y^{***}$ is the Dunford operator of I from Corollary 3.4:

$$(Df)(y^{**}) = \int_{B_{X^*} \times B_{Y^*}} f(x^*, y^*) y^{**}(y^*) \mathrm{d}\mu, \quad y^{**} \in Y^{**}.$$

Since $\ell_1 \not\hookrightarrow Y$, I is Pettis integrable, whence D takes values in Y^*. Moreover, since all Radon measures are perfect, D is compact by [15, Theorem 4-1-6]. Since $L_\infty(\mu)^*$ has the AP, given $\varepsilon > 0$, we can find some $F \in \mathcal{F}(L_\infty(\mu), Y^*)$ such that $\|D - F\| < \varepsilon$ [12, Theorem 1.e.5]. Consequently $\|T - (F \circ S)\| < \varepsilon \|S\| = \varepsilon$.

(ii) $\Rightarrow$ (i). If $\ell_1 \hookrightarrow Y$ then $L_1[0, 1] \hookrightarrow Y^*$ by Pełczyński's theorem [13]. The formal identity map from $L_\infty[0, 1]$ to $L_1[0, 1]$ is an integral operator. It is not compact because the Rademacher functions $\{r_n\}_{n=0}^\infty$ do not form a relatively norm compact subset of $L_1[0, 1]$. By taking compositions, we obtain an integral operator from $L_\infty[0, 1]$ to Y^* which is again non-compact. $\square$

We remark that the duality assumption on Y^* cannot be removed. Indeed, $\iota : L_\infty[0, 1] \to L_1[0, 1] \to c_0$, where $\iota : f \mapsto \int f r_n \mathrm{d}t$, is a factorization which shows that ι is an integral operator. But again, it is not compact.

4. The BAP in duals

In the last part of our note we give a new proof of another classical result of Grothendieck. The proof simply combines two dualities for tensor products.

Theorem 4.1 (*Grothendieck*)**.** *Let X be a dual Banach space with the RNP. Then X has the 1-BAP (i.e. MAP) whenever X has the AP.*

Proof. Let Y be a Banach space, $X = Y^*$ be its dual with the AP, and $z \in X^* \otimes_\pi X$. By Proposition 1.1 we have $(X^* \otimes_\pi X)^* = \mathcal{L}(X^*)$, so

$$\pi(z) = \sup_{\|T\| \le 1, T \in \mathcal{L}(X^*)} \langle T, z \rangle \ge \sup_{\|T\| \le 1, T \in \mathcal{L}(X)} \langle T^*, z \rangle. \tag{13}$$

On the other hand, since Y also has the AP, we have $\mathcal{K}(Y) = \overline{\mathcal{F}}(Y) = X \otimes_\varepsilon Y$ by Theorem 1.2 (vi). Thus, by Theorems 3.8 and 1.2(iv), we have $\mathcal{K}(Y)^* = \mathcal{N}(X) = X^* \otimes_\pi X$. Consequently,

$$\pi(z) = \sup_{\|T\| \le 1, T \in \mathcal{K}(Y)} \langle z, T \rangle \le \sup_{\|T\| \le 1, T \in \mathcal{K}(X)} \langle T^*, z \rangle = \sup_{\|T\| \le 1, T \in \mathcal{F}(X)} \langle T^*, z \rangle \tag{14}$$

The last equality follows from Theorem 1.2 (vi), since X has the AP. Combining (13) with (14), we obtain

$$\pi(z) = \sup_{\|T\|\le 1, T\in\mathcal{L}(X)} \langle T^*, z\rangle = \sup_{\|T\|\le 1, T\in\mathcal{F}(X)} \langle T^*, z\rangle. \tag{15}$$

Given $z = \sum_{i=1}^{\infty} x_i^* \otimes x_i \in X^* \otimes_\pi X$ and $T \in \mathcal{L}(X)$, we have the equality

$$\langle T^*, z\rangle = \sum_{i=1}^{\infty} \langle T^*(x_i^*), x_i\rangle = \sum_{i=1}^{\infty} \langle x_i^*, T(x_i)\rangle = i^*(z)(T)$$

where the mapping $i^* : X^* \otimes_\pi X \to (\mathcal{L}(X), \tau)^*$ is surjective, by Theorem 2.3. Thus by applying the Hahn–Banach theorem to $B_{\mathcal{F}(X)} \subset (\mathcal{L}(X), \tau)$ and using (15), we see that $I \in \overline{B_{\mathcal{F}(X)}}^{\tau}$, giving the 1-BAP. □

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References

[1] R.D. Bourgin, Geometric Aspects of Convex Sets with the Radon–Nikodým Property, in: LNM, vol. 993, Springer Verlag, 1983.

[2] A. Defant, K. Floret, Tensor Norms and Operator Ideals, in: North-Holland Math. Studies, vol. 176, 1993.

[3] J. Diestel, J.J. Uhl, Vector Measures, in: Mathematical Surveys, vol. 15, AMS, 1977.

[4] P. Enflo, A counterexample to the approximation problem in Banach spaces, Acta Math. 130 (1973) 309–317.

[5] M. Fabian, P. Habala, P. Hájek, V. Montesinos, V. Zizler, Banach Space Theory: The Basis for Linear and Nonlinear Analysis, in: CMS Books in Mathematics, Springer, 2011.

[6] T. Figiel, W.B. Johnson, The approximation property does not imply the bounded approximation property, Proc. Amer. Math. Soc. 41 (1973) 197–200.

[7] A. Grothendieck, Resume de la theorie metrique des produits tensoriels topologiques, Bol. Soc. Mat. Sao Paolo 8 (1953) 1–73.

[8] R.C. James, Separable conjugate spaces, Pacific J. Math. 10 (1960) 563–571.

[9] H. Jarchow, Locally Convex Spaces, Teubner, 1981.

[10] K. John, On the compact nonnuclear operator problem, Math. Ann. 287 (1990) 509–514.

[11] J. Lindenstrauss, On James' paper "separable conjugate spaces", Israel J. Math. 9 (1971) 279–284.

[12] J. Lindenstrauss, L. Tzafriri, Classical Banach Spaces I, Springer-Verlag, 1977.

[13] A. Pełczyński, On Banach spaces containing $L_1(\mu)$, Studia Math. 30 (1968) 231–246.

[14] H.P. Rosenthal, Some new characterizations of Banach spaces containing ℓ_1, http://arxiv.org/abs/0710.5944.

[15] M. Talagrand, Pettis Integral and Measure Theory, in: Memoirs AMS, vol. 307, 1984.